

How particles emerge from decaying classical fields in heavy ion collisions: towards a kinetic description of the Glasma

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Abstract

We develop the formalism discussed previously in hep-ph/0601209 and hep-ph/0605246 to construct a kinetic theory that provides insight into the earliest “Glasma” stage of a high energy heavy ion collision. Particles produced from the decay of classical fields in the Glasma obey a Boltzmann equation whose novel features include an inhomogeneous source term and new contributions to the collision term. We discuss the power counting associated with the different terms in the Boltzmann equation and outline the transition from the field dominated regime to the particle dominated regime in high energy heavy ion collisions.

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1 Introduction

In two previous papers [1,2], we introduced a formalism to compute multi-particle production in field theories coupled to strong time-dependent external sources. The QCD example of such a field theory is the Color Glass Condensate (CGC) [3–16]. For simplicity, we considered a ϕ^3 theory; we believe however that most of our results are of general validity and can be extended to gauge theories [17].

In this paper, we will address a problem in multi-particle production that was not considered in Refs. [1,2]. Specifically, the approach developed there did not include scattering processes that are important for the dynamics of the system at late times. These are the so called *secular terms* which are of higher order in the coupling constant (loop corrections) and are accompanied by growing powers of time [18–20]. The secular contributions must be resummed to obtain sensible results. In a quantum field theory, this resummation is performed in principle by solving the Dyson-Schwinger equations. In practice, the Dyson-Schwinger equations are difficult to solve. For a system of fields coupled to an ensemble of particles, it is well known that the Dyson-Schwinger equations can be approximated by a Boltzmann equation for the distribution of particles. The goal of the present paper is to extend the approach of Refs. [1,2] to derive a kinetic equation that includes the late time contributions to multi-particle production in field theories with strong external sources. We have in mind the dynamics after a heavy ion collision, where the classical field produced by the colliding nuclei expands rapidly into the vacuum along the beam direction. Our approach may also be of relevance to descriptions of the decay of the inflaton field and thermalization in the preheating and reheating phases of the early universe—a nice review with relevant references can be found in Ref. [21]. In both cases, as the classical field evolves, the occupation number decreases and it is more appropriate to describe the higher momentum modes of the system in terms of particle degrees of freedom.

The connections between the classical approximation in field theory and kinetic equations in the framework of nuclear collisions were previously discussed by Mueller and Son [22], and subsequently by Jeon [23]. They considered a system of fields in the presence of an ensemble of particles described by a distribution f . Performing a classical approximation in the path integral describing the evolution of this system and a gradient expansion in the obtained Dyson-Schwinger equations, these authors obtained a kinetic equation for f . An obvious question arises: with what accuracy does this kinetic equation reproduce the Boltzmann equation one would obtain without performing the classical approximation? The authors of Refs. [22,23] find that the kinetic equation obtained from the classical path integral reproduces correctly the collision term in the Boltzmann equation to leading power of f and (surprisingly) the first subleading term in f as well.

We shall adopt a more *ab initio* approach here by considering a system that does not contain any particle degrees of freedom *initially*, but where the fields are coupled to a strong time-dependent external source j . The external source is assumed to be a stochastic variable that belongs to an ensemble of charges specified by a distribution $W[j]$. This is the typical set up in the description of heavy ion collisions in the Color Glass Condensate framework where $W[j]$ represents the distribution of color charges. Because of the expansion of the system, one may anticipate that the system can be described by field theory methods at early times and by kinetic theory and hydrodynamics at later times. The matter in this regime in heavy ion collisions has interesting properties; two noteworthy possibilities are dynamically generated topological charge [24,25] and plasma

instabilities possibly leading to turbulent color fields [26]. This matter has been called an Glasma [27,28] and understanding its dynamical evolution holds the key to a deeper understanding of the strongly interacting Quark Gluon Plasma (sQGP) that may be formed at later times [29]. The

We will address here general questions about the dynamical evolution of such matter in the simplest possible context of a scalar (ϕ^3) field theory ¹:

- i. *What is the kinetic equation one obtains in field theories coupled to strong external sources?* Knowing the answer to this question is important for one to handle correctly the transition region between a field theory description and kinetic theory. Indeed, one expects from the work in Refs. [22,23] that there exists a window in time where both approaches correctly describe the dynamics ². This suggests that the kinetic equation in the overlap regime must know about the coupling of sources to fields at earlier times. How is this manifest, how important is this effect and how does it go away?
- ii. *What terms in the kinetic equation are important at different stages of the expansion?* The previous question hints that we will obtain a kinetic equation that has additional terms absent in the conventional Boltzmann equation. We would like to understand how this generalized Boltzmann equation converges to the usual one at late times.

The paper is organized as follows. In section 2, we shall remind the reader of relevant formulae in the derivation [1] of the average number $\langle n \rangle$ of produced particles. In section 3, we shall write down the Dyson-Schwinger equations for the two-point functions in theories with time dependent strong sources. These provide the starting point for a derivation in section 4 of the corresponding kinetic equation for the Glasma. We observe that the coupling of the field to an external source leads to an inhomogeneous term in this kinetic equation. In section 5, we discuss the properties of the different terms appearing in the kinetic equation. Albeit the collision term in the kinetic equation looks identical to the collision term in the usual Boltzmann equation, it contains novel contributions to the self energy that are of 0-loop and 1-loop order. We discuss the power counting for these different contributions and assess their relative contribution at different stages of the temporal evolution of the Glasma. We conclude with a brief summary and outlook emphasizing unresolved issues. An appendix addresses how the averaging over the sources j in our formalism can be re-expressed in terms of the usual ensemble average implicit in the derivation of kinetic equations.

¹Even the “simple” scalar theory is non-trivial. It will indeed contain very general features of relevance to the Glasma albeit the latter will have significant (and very interesting) additional features that are absent in the scalar case.

²This has to be the case if one wants the final result to be independent of the time at which one switches between the two descriptions.

2 *Ab initio* computation of $\langle n \rangle$

We consider the theory of a real scalar field ϕ with cubic self-interactions, coupled to an external time dependent source $j(x)$. The Lagrangian of the model is

$$\mathcal{L} \equiv \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{g}{3!} \phi^3 + j\phi. \quad (1)$$

In [1], we systematically calculated particle production from these sources. In the Color Glass Condensate framework that this toy model mimics, the colliding projectiles are represented by a statistical ensemble of currents j . Physical quantities are obtained by averaging over all possible realizations of the j 's. In this section, we shall discuss the calculation of the average number of produced particles in a given configuration of j 's.

A general formula for the average number $\langle n \rangle$ of produced particles is

$$\langle n \rangle = \int \frac{d^3 \mathbf{p}}{(2\pi)^3 2E_p} \langle 0_{\text{in}} | a_{\text{out}}^\dagger(\mathbf{p}) a_{\text{out}}(\mathbf{p}) | 0_{\text{in}} \rangle. \quad (2)$$

The number of particles produced with a certain momentum \mathbf{p} is defined as the expectation value of the “out” number operator in the initial state. This formula gives the number of particles at asymptotic times, after the particles have decoupled³.

A simple reduction formula gives [30]

$$\begin{aligned} \langle 0_{\text{in}} | a_{\text{out}}^\dagger(\mathbf{p}) a_{\text{out}}(\mathbf{p}) | 0_{\text{in}} \rangle &= \frac{1}{Z} \int d^4 x d^4 y e^{-ip \cdot x} e^{ip \cdot y} \\ &\times (\square_x + m^2)(\square_y + m^2) \langle 0_{\text{in}} | \phi(x) \phi(y) | 0_{\text{in}} \rangle, \end{aligned} \quad (3)$$

where Z is the wave function renormalization factor. The expectation value in the right hand side of this equation has two important features : (i) the vacuum state is the “in” vacuum state on both sides and, (ii) the two fields inside the correlator are not time-ordered. The Schwinger-Keldysh formalism [31,32] provides techniques for computing these types of correlators.

The operators $\square + m^2$ amputate the external legs of the two-point function $G_{-+}(x, y) \equiv \langle 0_{\text{in}} | \phi(x) \phi(y) | 0_{\text{in}} \rangle$. Defining

$$\tilde{G}_{-+}(x, y) \equiv \frac{(\square_x + m^2)(\square_y + m^2)}{Z} G_{-+}(x, y), \quad (4)$$

we can write the average multiplicity as

$$\langle n \rangle = \int \frac{d^3 \mathbf{p}}{(2\pi)^3 2E_p} \int d^4 x d^4 y e^{-ip \cdot x} e^{ip \cdot y} \tilde{G}_{-+}(x, y). \quad (5)$$

Introducing the variables

$$X \equiv \frac{x+y}{2}, \quad r \equiv x-y, \quad (6)$$

³The “number of particles” at some intermediate time, while the fields are still interacting, is not a well defined concept.

we can rewrite this formula as

$$E_{\mathbf{p}} \frac{d\langle n \rangle}{d^3 \mathbf{p}} = \frac{1}{16\pi^3} \int d^4 X \tilde{G}_{-+}(X, p), \quad (7)$$

where

$$\tilde{G}_{-+}(X, p) \equiv \int d^4 r e^{-ip \cdot r} \tilde{G}_{-+} \left(X + \frac{r}{2}, X - \frac{r}{2} \right) \quad (8)$$

is the Wigner transform of $\tilde{G}_{-+}(x, y)$.

In the Schwinger–Keldysh formalism, the propagators $\mathbf{G}_{\epsilon\epsilon'}(x, y)$, $(\epsilon, \epsilon' = +, -)$ can be expressed as

$$\mathbf{G}_{\epsilon\epsilon'}(x, y) = \frac{\delta}{i\delta j_{\epsilon}(x)} \frac{\delta}{i\delta j_{\epsilon'}(y)} e^{i\mathcal{V}_{SK}[j_+, j_-]} \Big|_{j_+=j_-=j}, \quad (9)$$

where $i\mathcal{V}_{SK}[j_+, j_-]$ is the sum of all **connected** vacuum-vacuum diagrams. When $j_+ = j_- = j$, $i\mathcal{V}_{SK}[j, j] = 0$ and the sum of all vacuum-vacuum diagrams is unity.

Working out the functional derivatives,

$$\mathbf{G}_{\epsilon\epsilon'}(x, y) = \left[\frac{\delta i\mathcal{V}_{SK}}{i\delta j_{\epsilon}(x)} \frac{\delta i\mathcal{V}_{SK}}{i\delta j_{\epsilon'}(y)} + \frac{\delta^2 i\mathcal{V}_{SK}}{i\delta j_{\epsilon}(x) i\delta j_{\epsilon'}(y)} \right]_{j_+=j_-=j}. \quad (10)$$

As $i\mathcal{V}_{SK}$ is the sum of connected vacuum-vacuum diagrams, any of its deriva-

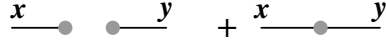


Figure 1: Diagrammatic representation of the disconnected (left) and connected (right) terms in eq. (10). The gray blobs denote the remnants of Green's functions after the free propagators at the endpoints are amputated.

tives with respect to j_{\pm} is a connected Green's function. Therefore, $\mathbf{G}_{\epsilon\epsilon'}$ can be decomposed as

$$\mathbf{G}_{\epsilon\epsilon'}(x, y) \equiv \mathbf{G}_{\epsilon\epsilon'}^c(x, y) + \mathbf{G}_{\epsilon\epsilon'}^{\text{nc}}(x, y). \quad (11)$$

These are, respectively, the connected part

$$\mathbf{G}_{\epsilon\epsilon'}^c(x, y) \equiv \frac{\delta^2 i\mathcal{V}_{SK}}{i\delta j_{\epsilon}(x) i\delta j_{\epsilon'}(y)} \Big|_{j_+=j_-=j}, \quad (12)$$

and a disconnected part corresponding to the product of the expectation values of the field at the points x and y :

$$\mathbf{G}_{\epsilon\epsilon'}^{\text{nc}}(x, y) = \langle \phi(x) \rangle \langle \phi(y) \rangle \quad \text{with} \quad \langle \phi(x) \rangle = \frac{\delta i\mathcal{V}_{SK}}{i\delta j_{\pm}(x)} \Big|_{j_+=j_-=j}. \quad (13)$$

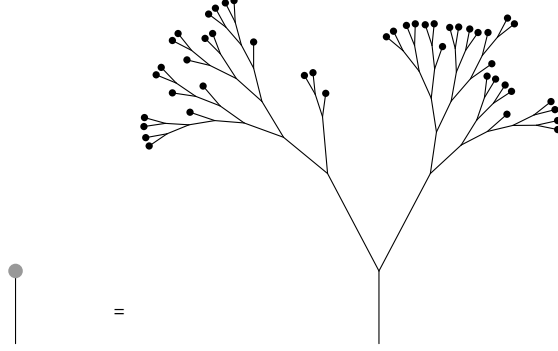


Figure 2: Example of a tree diagram contributing to the field expectation value. The black dots terminating branches of the tree represent insertions of the source j in the diagram on the right. The sum of these tree diagrams is represented (left) by a line attached to a gray blob.

When $j_+ = j_- = j$, the expectation value of the field is the same on the upper and lower branches of the contour: $\langle \phi_+(x) \rangle = \langle \phi_-(x) \rangle$. This explains why we omitted the $+/-$ index in the expectation value of the field.

A typical tree-level contribution to $\langle \phi(x) \rangle$ is shown in figure 2. Note also that $\langle \phi(x) \rangle$ vanishes if the external source $j(x)$ is zero⁴. At tree level, because $j_+ = j_- = j$, the sum over the $+/-$ indices in the Schwinger-Keldysh formalism at all the internal vertices of the tree (including the sources) can be performed by using the identities

$$\mathbf{G}_{++}^0 - \mathbf{G}_{+-}^0 = \mathbf{G}_R^0 \quad , \quad \mathbf{G}_{-+}^0 - \mathbf{G}_{--}^0 = \mathbf{G}_R^0 \quad , \quad (14)$$

where \mathbf{G}_R^0 is the free retarded propagator⁵. When this sum is performed, all propagators in the tree diagram can be simply replaced by retarded propagators. This is equivalent to the statement that $\langle \phi(x) \rangle$ is the retarded solution of the classical equation of motion,

$$(\square + m^2)\phi(x) + \frac{g}{2}\phi^2(x) = j(x) \quad , \quad (15)$$

with a vanishing boundary condition at $x_0 = -\infty$.

Eq. (7) is the complete answer to the problem of particle production in the effective theory described by the Lagrangian of eq. (1). If one were able to compute $\tilde{G}(x, y)$ to all orders, this formula would contain everything one needs. There would be no need for tools such as kinetic theory.

⁴We assume that the self-interactions of the fields are such that there is no spontaneous breakdown of symmetry when $j = 0$.

⁵In momentum space, this propagator reads $\mathbf{G}_R^0(p) = i/(p^2 - m^2 + ip^0\epsilon)$.

However, evaluating eq. (7) to all orders is an unrealistic goal. What has been implemented thus far is the evaluation of eq. (7) at leading order (tree level) to calculate the gluon yield in high-energy nucleus-nucleus collisions [33–38]. In [1], an algorithm was sketched to compute $\langle n \rangle$ at next-to-leading order (one loop) in terms of the retarded classical field and of retarded fluctuations propagating in the classical field background.

In practice, one has to truncate the loop expansion. As we will discuss in the next section, the correct way to perform practical calculations is within the framework of the Dyson–Schwinger equations.

3 Dyson-Schwinger equations

The main problem with the loop expansion described in the previous section is that, in general, truncations in $\tilde{G}(x, y)$ will lead to an incorrect large time limit of the number of produced particles. This can be traced to secular terms containing powers of the time that invalidate the perturbative series in the large time limit. This can be cured by appropriate resummation; the well known way to do this is to solve Dyson-Schwinger equations [18–20]. In this section, we shall discuss the Dyson-Schwinger equations obeyed by the two-point functions $G_{\epsilon\epsilon'}(x, y)$ of the Schwinger-Keldysh formalism. We will see that the presence of a disconnected contribution to these 2-point functions leads to interesting features in the corresponding Dyson-Schwinger equations.

3.1 Dyson-Schwinger equation for the connected part

It is straightforward to write a Dyson-Schwinger equation for the connected part of the 2-point function, $G_{\epsilon\epsilon'}^c$, that resums self-energy corrections :

$$G^c(x, y) = G^0(x, y) + \int_{\mathcal{C}} d^4u d^4v G^0(x, u) \left[-i\Sigma(u, v) \right] G^c(v, y) , \quad (16)$$

where $-i\Sigma$ is a 1-particle irreducible connected⁶ self-energy, *evaluated in the presence of external sources*. We shall not write here explicitly the \pm indices carried by the various objects. Instead, we write the time integrations as integrals over the complete Schwinger-Keldysh contour \mathcal{C} .

It is convenient to extract from this self-energy a local piece, by writing

$$\Sigma(u, v) \equiv g\Phi(u)\delta(u - v) + \Pi(u, v) . \quad (17)$$

Except for the background field, which is a genuine local contribution to the self-energy, there is a certain arbitrariness in this separation because it depends on the momentum scale at which we resolve the system. A contribution to the self-energy that does not change significantly over space-time scales on the order of the Compton wavelength p^{-1} can be treated as a mean field at that scale.

⁶It is connected in order to have a connected 2-point function after the resummation and it needs to be 1PI to prevent double counting.

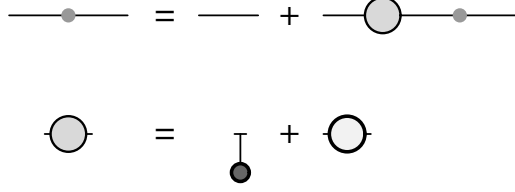


Figure 3: Top: Diagrammatic representation of the Dyson-Schwinger equation of eq. (16). The large gray blob denotes the 1-particle irreducible 2-point function Σ . Bottom: decomposition of the 1PI self-energy Σ into a local part $g\Phi$ and a non-local part Π (denoted by a large light-gray blob), following eq. (17).

Therefore, the mean field term $\Phi(u)$ will contain the classical field, and possibly changes in the dispersion relation due to medium effects ⁷.

The Dyson-Schwinger equation then becomes

$$\begin{aligned} \mathbf{G}^c(x, y) &= \mathbf{G}^0(x, y) - ig \int_{\mathcal{C}} d^4u \mathbf{G}^0(x, u) \Phi(u) \mathbf{G}^c(u, y) \\ &\quad + \int_{\mathcal{C}} d^4u d^4v \mathbf{G}^0(x, u) [-i\Pi(u, v)] \mathbf{G}^c(v, y) . \end{aligned} \quad (18)$$

Using

$$[\Box_x + m^2] \mathbf{G}^0(x, y) = -i\delta_c(x - y) , \quad (19)$$

where δ_c denotes the delta function on the closed time path⁸, we can rewrite this equation as

$$[\Box_x + m^2 + g\Phi(x)] \mathbf{G}^c(x, y) = -i\delta_c(x - y) - \int_{\mathcal{C}} d^4u \Pi(x, u) \mathbf{G}^c(u, y) . \quad (20)$$

3.2 Dyson-Schwinger equation for the disconnected part

We also need a Dyson-Schwinger equation for the disconnected part of the Green's function,

$$\mathbf{G}^{\text{nc}}(x, y) = \langle \phi(x) \rangle \langle \phi(y) \rangle . \quad (21)$$

Because the expectation value $\langle \phi \rangle$ is a connected 1-point function, it is natural to factor the connected propagator out of it, by writing

$$\langle \phi(x) \rangle \equiv \int_{\mathcal{C}} d^4u \mathbf{G}^c(x, u) \mathbf{S}(u) , \quad (22)$$

⁷To allow for this possibility, we denote the mean field piece by a symbol distinct from the one used for the classical field.

⁸ $\delta_c(x - y) = 0$ unless x^0 and y^0 are equal and lie on the same branch of the time path.

where $\mathcal{S}(u)$ is an “effective source” term⁹. By construction, one obtains

$$[\square_x + m^2 + g\Phi(x)]\langle\phi(x)\rangle = -i\mathcal{S}(x) - \int_{\mathcal{C}} d^4u \, \Pi(x, u)\langle\phi(u)\rangle . \quad (23)$$

Multiplying both sides by $\langle\phi(y)\rangle$, one obtains

$$[\square_x + m^2 + g\Phi(x)]\mathcal{G}^{\text{nc}}(x, y) = -i\mathcal{S}(x)\langle\phi(y)\rangle - \int_{\mathcal{C}} d^4u \, \Pi(x, u)\mathcal{G}^{\text{nc}}(u, y) . \quad (24)$$

Defining

$$-i\Pi^{\mathcal{S}}(x, y) \equiv \mathcal{S}(x)\mathcal{S}(y) , \quad (25)$$

we can rewrite this equation as

$$[\square_x + m^2 + g\Phi(x)]\mathcal{G}^{\text{nc}}(x, y) = - \int_{\mathcal{C}} d^4u \left[\Pi^{\mathcal{S}}(x, u)\mathcal{G}^{\text{c}}(u, y) + \Pi(x, u)\mathcal{G}^{\text{nc}}(u, y) \right] . \quad (26)$$

Adding eqs. (20) and (26), we obtain the Dyson-Schwinger equation for the complete two-point function:

$$\begin{aligned} [\square_x + m^2 + g\Phi(x)]\mathcal{G}(x, y) &= -i\delta_{\mathcal{C}}(x - y) \\ &\quad - \int_{\mathcal{C}} d^4u \left[\Pi^{\mathcal{S}}(x, u)\mathcal{G}^{\text{c}}(u, y) + \Pi(x, u)\mathcal{G}(u, y) \right] . \end{aligned} \quad (27)$$

The only formal difference between this Dyson-Schwinger equation and the equation one obtains in the absence of the source j is the term proportional to $\Pi^{\mathcal{S}}$ in the right hand side.

In principle, the resummations performed by solving eqs. (20) and (26) (or, equivalently, eqs. (20) and (27)) would completely cure the problem of secular terms. Such an approach has been pursued numerically in [39], but has not been attempted yet in the context of heavy ion collisions in the CGC framework.

4 Kinetic equation

The Dyson-Schwinger equations we wrote down in the previous section contain all the necessary physics but their solution is likely too difficult; they therefore by themselves do not provide any practical insight into the dynamics of high energy heavy ion collisions. One can simplify the problem a step further by transforming the Dyson-Schwinger equations for the 2-point functions into kinetic equations. However, as we shall discuss shortly, doing so requires that certain assumptions be satisfied.

⁹In the classical limit, one has $\mathcal{S}(x) = j(x) + \frac{g}{2}\phi^2(x)$ (see section 5.2).

4.1 Fields and particles

As is well known, the Boltzmann kinetic equation describes the space-time evolution of particle phase space densities. Therefore, to achieve a kinetic description, the formalism considered thus far should be extended to incorporate an ensemble of particles. This is simply done by modifying the free propagators to add a term that depends on the distribution of particles $f(\mathbf{p})$. In momentum space, the modified propagators are¹⁰

$$\begin{aligned} G_{++}^0(p) &\equiv \frac{i}{p^2 - m^2 + i\epsilon} + 2\pi f(\mathbf{p})\delta(p^2 - m^2) , \\ G_{--}^0(p) &\equiv \frac{-i}{p^2 - m^2 - i\epsilon} + 2\pi f(\mathbf{p})\delta(p^2 - m^2) , \\ G_{-+}^0(p) &\equiv 2\pi(\theta(p^0) + f(\mathbf{p}))\delta(p^2 - m^2) , \\ G_{+-}^0(p) &\equiv 2\pi(\theta(-p^0) + f(\mathbf{p}))\delta(p^2 - m^2) . \end{aligned} \quad (28)$$

These modified rules for the Schwinger-Keldysh propagators can be derived [40] when the *initial* density matrix that describes the ensemble has the form

$$\rho \equiv \exp \left[- \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}} \beta_{\mathbf{p}} E_{\mathbf{p}} a_{\text{in}}^\dagger(\mathbf{p}) a_{\text{in}}(\mathbf{p}) \right] , \quad (29)$$

where $\beta_{\mathbf{p}}$ is a momentum dependent quantity. (Note: $\beta_{\mathbf{p}}$ should not be confused with the inverse temperature.) Such a form for the density matrix is required if correlators computed with this density matrix are to satisfy Wick's theorem. From this form of the density matrix, one obtains the Schwinger-Keldysh rules of eqs. (28), with

$$f(\mathbf{p}) = \frac{1}{e^{\beta_{\mathbf{p}} E_{\mathbf{p}}} - 1} . \quad (30)$$

The function $f(\mathbf{p})$ in the propagators only represents the *initial* distribution of particles in the system. Thus the field theory defined by the Lagrangian of eq. (1) and the propagators of eqs. (28) describes a system of fields coupled to an external source j and to an ensemble of particles with an initial distribution $f(\mathbf{p})$. The Feynman rules then enable one to calculate the properties of this system at a later time.

However, eqs. (28) do not lead to a well behaved perturbative expansion, except when the function $f(\mathbf{p})$ is the equilibrium Bose-Einstein distribution in our model of bosonic fields. In general, when $f(\mathbf{p})$ is not a Bose-Einstein distribution, the perturbative expansion based on eqs. (28) is plagued by the previously mentioned pathological secular terms which need to be resummed. The time-scale at which resummation becomes necessary is related to the transport mean free path in the system, namely, the time between two large angle scatterings undergone by a particle. This resummation makes the distribution $f(\mathbf{p})$ time-dependent reflecting the changes induced by collisions on the particle

¹⁰The propagators of the Schwinger-Keldysh formalism appropriate for calculating eq. (3) are the same with $f(\mathbf{p}) = 0$.

phase space distribution. Under certain approximations to be discussed later, this temporal evolution is governed by a Boltzmann equation.

The problem formulated in section 2 concerned a system that has no ensemble of particles at the initial time ($f(\mathbf{p}) = 0$ in eqs. (28)). At first sight, as $f = 0$ is a particular case of the Bose–Einstein distribution (with a vanishing temperature), secular divergences may appear to be absent. However, this conclusion is incorrect because of the presence of external sources which drive the system out of equilibrium. Thus it is also necessary to resum secular terms in this case, leading to changes in $f(\mathbf{p})$. The generalized propagators in eq. (28) constitute the natural framework to achieve this. Because the external source is both time and space dependent, one has more generally

$$f(\mathbf{p}) \rightarrow f(X, \mathbf{p}) \quad (31)$$

in eqs. (28).

An important point must be made here about the tree level expectation values $\langle \phi_{\pm}(x) \rangle$ in this f -dependent extension of our formalism. A crucial property of the propagators in eqs. (28) is that they still obey eqs. (14). The retarded propagator is therefore *f -independent*. Therefore, as long as loop corrections are not included, the field expectation value does not depend on f and is identical to the result obtained from the retarded solution of the classical equations of motion. Hence, the contribution from the disconnected part of the 2-point function lead to an inhomogeneous (f -independent) term in the Boltzmann equation.

4.2 Gradient expansion

The extension (28) of the propagators leads to Dyson-Schwinger equations that are formally identical to eqs. (20) and (26) – with all the building blocks now constructed with f -dependent propagators. The first step in obtaining the Boltzmann equation is to rewrite all the distributions in terms of their Wigner transforms. For a two-point function $F(x, y)$, its Wigner transform $\tilde{F}(X, p)$ is defined to be

$$\tilde{F}(X, p) \equiv \int d^4s \, e^{-ip \cdot s} F\left(X + \frac{s}{2}, X - \frac{s}{2}\right). \quad (32)$$

The next step is to perform a gradient expansion where only long wavelength, low momentum modes are retained. In particular, all terms of order two or higher in ∂_x are neglected. As our goal is to construct a kinetic theory for the Glasma, we will discuss the validity of this gradient expansion in the context of heavy ion collisions in the CGC framework. In this framework [14–16], the color sources $\rho^a(\mathbf{x}_{\perp})$ generating the color currents¹¹ are stochastic variables that vary from event to event with a distribution $W[\rho]$. When calculating a given physical quantity, one first computes it for an arbitrary ρ and then averages over all possible ρ 's in the ensemble generated with the weight $W[\rho]$. For example,

¹¹These color sources are the QCD analogs of the sources j in our toy scalar theory.

in the McLerran-Venugopalan model [3–5], the distribution $W[\rho]$ is a Gaussian with

$$W[\rho] = \exp \left(- \int d^2 \mathbf{x}_\perp d^2 \mathbf{y}_\perp \frac{\rho(\mathbf{x}_\perp) \rho(\mathbf{y}_\perp)}{2 \mu^2(\mathbf{x}_\perp, \mathbf{y}_\perp)} \right), \quad (33)$$

where

$$\mu^2(\mathbf{x}_\perp, \mathbf{y}_\perp) \equiv \langle \rho(\mathbf{x}_\perp) \rho(\mathbf{y}_\perp) \rangle = \mu_A^2(\mathbf{x}_\perp) \delta(\mathbf{x}_\perp - \mathbf{y}_\perp). \quad (34)$$

Here $\mu_A^2(\mathbf{x}_\perp)$ represents the density of color charges at a spatial position \mathbf{x}_\perp in the nucleus. The typical momentum scale of the sources—the saturation momentum squared Q_s^2 at \mathbf{x}_\perp is simply related to μ_A^2 .

The difference between one particular element of the ensemble and the average weighted by $W[\rho]$ is illustrated in figure 4 for the quadratic form ρ^2 . Because

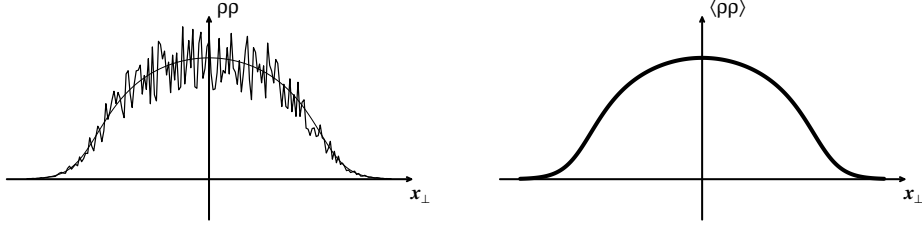


Figure 4: Left: ρ^2 distribution for one configuration in the ensemble represented by the distribution $W[\rho]$. Right: ensemble average of $\langle \rho^2 \rangle$.

the $\rho(\mathbf{x}_\perp)$ are uncorrelated at different points in the transverse plane of the nucleus, a particular configuration of ρ 's leads to a very rough density profile; in contrast, the average smoothly follows the Woods-Saxon density profile of a nucleus. This example simply illustrates that the gradients are uncontrollably large for a given configuration ρ rendering any gradient expansion meaningless. On the other hand, it is perfectly legitimate for ensemble averaged quantities.

The typical momenta of “hard” particles is set by the saturation scale which is of order $Q_s \sim 1\text{--}2$ GeV at RHIC energies; this scale may be higher at the LHC. In contrast, the gradient ∂_x for averaged quantities changes appreciably over distance scales of the inverse nuclear radius given by $\sim R_A^{-1} \sim 40$ MeV for a large nucleus. The small magnitude of this scale in the gradient expansion relative to the typical saturation momentum justifies the gradient expansion for quantities that are averaged over the ensemble of color charges.

The corresponding changes to the Feynman rules are described in appendix A. Here it is sufficient to note that the ensemble average is obtained by connecting all the external sources j in the manner specified by the distribution $W[\rho]$. For instance, in the MV model $W[\rho]$ is a Gaussian, which implies that all the sources must be connected pairwise. The objects Π^S , Π and Φ that appear in the Dyson-Schwinger equations (20) and (26) must be thought of as being averaged over j . In Feynman diagrams, we will represent the average over j by

surrounding the diagram by a light gray halo :

$$\langle \langle \phi(x) \rangle \rangle_j = \text{[Diagram: a vertical line with a gray-shaded circle at the top]} . \quad (35)$$

This compact notation encompasses a very large number of contributions. For instance, at leading order, one would first approximate $\langle \phi(x) \rangle$ as the sum of all the tree diagrams, an example of which is represented in figure 2. For each such tree diagram, the sources j (the black dots in figure 2) are reconnected pairwise in all the possible ways. A typical reconnection of the sources, corresponding to the topology of figure 2, is displayed in figure 5. Note that the “loop order”

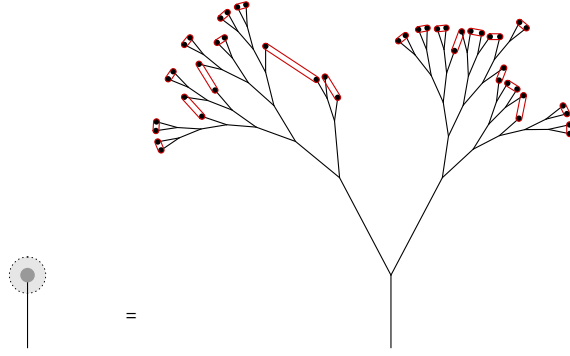


Figure 5: Example of a tree level contribution to the average over the sources j of the field expectation value for a Gaussian distribution of sources. The links in red represent the elementary correlators $\langle j(x)j(y) \rangle$. The source connections represented here are for simplicity among nearest neighbors; all other pairwise topologies are feasible.

of a given diagram is a meaningful concept only for diagrams *before* they are averaged over j . Indeed, as one can see by comparing the figures 2 and 5, the diagram before the j -average has 0 loops and is of order g^{-1} . After the average is performed, while it has a large number of “loops” which do not contain any information about the order in g of the diagram.

4.3 Boltzmann equation

The final ingredient in the derivation of the Boltzmann equation is the so-called “quasi-particle ansatz” which can be expressed as

$$\begin{aligned} G_{-+}(X, p) &= (1 + f(X, \mathbf{p}))\rho(X, p) , \\ G_{+-}(X, p) &= f(X, \mathbf{p})\rho(X, p) , \end{aligned} \quad (36)$$

where the spectral function $\rho(X, p)$ is

$$\rho(X, p) \equiv \mathbf{G}_R(X, p) - \mathbf{G}_A(X, p) = \mathbf{G}_{-+}(X, p) - \mathbf{G}_{+-}(X, p). \quad (37)$$

The physical assumption here is that the interactions in the system are such that the collisional width of the dressed particles remains small compared to their energy; the system is made up of long-lived quasi-particles.

The Boltzmann equation can now be obtained as follows:

- i. Write a Dyson-Schwinger equation analogous to eq. (27), but with the differential operator $\square + m^2 + g\Phi$ acting on the variable y instead of x , and subtract it from eq. (27).
- ii. Rewrite this equation in terms of the Wigner transformed quantities and perform a gradient expansion keeping only leading terms in ∂_x .
- iii. Replace the Green's functions with the quasi-particle ansatz and drop the spectral function $\rho(X, p)$ which appears as a factor in all the terms.

If the terms proportional to Π^S were absent from eq. (27), the steps outlined above would result in the well-known Boltzmann-Vlasov equation,

$$\begin{aligned} 2p \cdot \partial_x f(X, \mathbf{p}) + g \partial_x \Phi(X) \cdot \partial_p f(X, \mathbf{p}) &= \\ &= (1 + f(X, \mathbf{p})) \Pi_{+-}(X, p) - f(X, \mathbf{p}) \Pi_{-+}(X, p). \end{aligned} \quad (38)$$

The extra term we have in the Dyson-Schwinger equations, proportional to Π^S , will modify the Boltzmann-Vlasov equation. Two key features of this novel term will prove essential in our derivation. The first is that $\Pi^S(x, y)$ does not depend on whether the points x and y are on the upper or lower branch of the time contour. This is because the expectation value of the field, for equal values of the sources j_+ and j_- , is the same on both branches of the contour. The second feature is that the non-connected part of the propagators drops out of the spectral function, for the same reason. Hence,

$$\rho(X, p) = \mathbf{G}_{-+}(X, p) - \mathbf{G}_{+-}(X, p) = \mathbf{G}_{-+}^c(X, p) - \mathbf{G}_{+-}^c(X, p). \quad (39)$$

Utilizing these two properties, we can perform the gradient expansion for this additional term in the same way as performed for the usual self-energy correction. It modifies the right hand side of the Boltzmann equation by an additive correction¹² $\Pi^S(X, p)$. Therefore, our final expression for the kinetic equation is

$$\begin{aligned} 2p \cdot \partial_x f(X, \mathbf{p}) + g \partial_x \Phi(X) \cdot \partial_p f(X, \mathbf{p}) &= \\ &= \Pi^S(X, p) + (1 + f(X, \mathbf{p})) \Pi_{+-}(X, p) - f(X, \mathbf{p}) \Pi_{-+}(X, p). \end{aligned} \quad (40)$$

¹²Note that prior to dropping the spectral function that appears in all terms, we would have

$$\begin{aligned} \Pi_{+-}^S(X, p) \mathbf{G}_{-+}^c(X, p) - \Pi_{-+}^S(X, p) \mathbf{G}_{+-}^c(X, p) &= \Pi^S(X, p) [\mathbf{G}_{-+}^c(X, p) - \mathbf{G}_{+-}^c(X, p)] \\ &= \Pi^S(X, p) \rho(X, p). \end{aligned}$$

The novel “source term” $\Pi^s(X, p)$ in this equation is non-zero even if the particle distribution $f(X, \mathbf{p})$ is zero. It is therefore responsible for $f = 0$ not being a fixed point of the above equation; the solution of this equation is non-zero at later times even if the initial condition had a vanishing particle distribution. In the next section, we will discuss further significant differences between this kinetic equation and the conventional Boltzmann-Vlasov equation in eq. (38).

5 Properties of the Glasma kinetic equation

In this section, we shall discuss the various terms in eq. (40) with emphasis on the differences between these and those appearing in the conventional Boltzmann kinetic equation.

5.1 Vlasov term

We first consider the Vlasov term $(g\partial_x \Phi \cdot \partial_p f)$ in the Boltzmann equation. We note that in performing the average of the *mean field* $\Phi(X)$, over the external sources j , the various correlation functions $\langle j(x_1) \cdots j(x_n) \rangle$ permitted by the distribution of sources $W[j]$ are nearly translation invariant. The dependence of these correlators on the barycentric co-ordinate $X \equiv (x_1 + \cdots + x_n)/n$ is very slow because it arises from the density profile of the colliding nuclei¹³. Therefore the 1-point function $\Phi(X)$, averaged over j , also has a very slow dependence on its argument X ; its Fourier transform with respect to X has only modes with momenta on the order of the inverse nuclear radius. As discussed previously, this scale is very small relative to the typical momentum of the particles under consideration and it is therefore legitimate to approximate it as a Vlasov term.

As is well known, the effect of this term in the Boltzmann equation is to change the momentum of particles as they move between regions where the external field is different. Indeed, $g\partial_x \Phi$ is the force that acts on the particles at point X and accelerates them towards regions of lower potential¹⁴. The mean field Φ includes not only the classical field directly produced by the external sources, but also possibly a contribution coming from the particles encoded in $f(X, \mathbf{p})$. Such a modification may arise from a modification of the particle dispersion relation due to the collective action of the other particles. For instance, if the particles acquire a medium mass with a weak space-time dependence, this mass can be represented by a potential in the Vlasov term of the kinetic equation.

¹³The fact that this density profile is not a constant is the only effect in the problem that breaks translation invariance.

¹⁴For non central collisions, the shape of the overlap region between the two nuclei is elliptic; one has stronger gradients in the direction of the small axis of the ellipsis relative to those in the direction of its large axis. The Vlasov term therefore accelerates particles preferentially in the direction of the small axis of the overlap region. This leads eventually to elliptic flow and to an anisotropy of the spectrum of particles in momentum space. This effect is obtained entirely within kinetic theory without any assumption about the degree of thermalization of the system.

5.2 Source term in the kinetic equation

Let us now consider the effect of the source term $\Pi^s(X, p)$ in eq. (40), which can be obtained as the Wigner transform of the product $\mathcal{S}(x)\mathcal{S}(y)$. An interesting situation, relevant for heavy ion collisions, is when tree diagrams are dominant because the external source is strong ($gj \sim 1$). In this case, the expectation value $\langle \phi(x) \rangle$ is dominated by the retarded classical field $\phi(x)$; the connected part of the 2-point function, \mathbf{G}^c , is simply the propagator of a fluctuation on top of the classical field,

$$(\mathbf{G}^c)^{-1} = \square + m^2 + g\phi . \quad (41)$$

One therefore immediately obtains the following expression¹⁵ for $\mathcal{S}(x)$:

$$\begin{aligned} \mathcal{S}(x) &= \left[\square + m^2 + g\phi(x) \right] \phi(x) \\ &= j(x) + \frac{g}{2}\phi^2(x) . \end{aligned} \quad (42)$$

We see here that the effective source $\mathcal{S}(x)$ receives two contributions :

- i. the external source $j(x)$ itself. This term is only important if we want to use the Boltzmann equation in regions of space-time where the external source is still active. In a heavy ion collision, the color sources are present only on the light-cone at a proper time $\tau = 0$. We will not consider this term further.
- ii. A term quadratic in the classical field produced by the external source; this term continues to contribute after the external sources have stopped acting.

One may represent this effective source graphically as

$$\mathcal{S}(x) \equiv \text{---}\otimes = \text{---}\bullet + \text{---}\begin{array}{c} \nearrow \\ \searrow \end{array} . \quad (43)$$

The second term has a fairly straightforward interpretation. When the term quadratic in ϕ in the classical equation of motion

$$\left[\square + m^2 \right] \phi(x) = j(x) - \frac{g}{2}\phi^2(x) , \quad (44)$$

is important, we see that the field is not a free field. If expanded in particle modes, the number of particles in the field would change with time. Therefore, if one switches between a description in terms of classical fields to the kinetic

¹⁵This is the result for a potential $g\phi^3/3!$. For an arbitrary potential $V(\phi)$, the expression of $\mathcal{S}(x)$ in this approximation would read

$$\mathcal{S}(x) = j(x) - V'(\phi(x)) + \phi(x)V''(\phi(x)) ,$$

where the prime denotes a derivative of the potential with respect to ϕ .

equation at a stage where this non-linear term is still significant, the source term in the Boltzmann equation modifies the number of particles in order to take this effect into account.

At tree level, the effective source $\mathbf{S}(x)$, and hence $\mathbf{\Pi}^s$, is independent of the distribution of particles f . As discussed previously, this is a straightforward consequence of the fact that, at tree level, the 1-point function in the Schwinger-Keldysh formalism can be rewritten entirely in terms of retarded propagators that are f -independent. $\mathbf{\Pi}^s$ is therefore non-zero even if $f = 0$. In contrast, the terms $\mathbf{\Pi}_{\pm\mp}$ in the r.h.s of the Boltzmann equation depend on f and vanish when $f = 0$ as expected for *collision terms*. $\mathbf{\Pi}^s$ is therefore a *source term* in the Boltzmann equation, because it drives f to a non-zero value even if one has $f = 0$ initially.

When we perform the average over j of the disconnected product $\mathbf{S}(x)\mathbf{S}(y)$, we get both disconnected and connected source terms,

$$\langle \mathbf{S}(x)\mathbf{S}(y) \rangle_j = \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} . \quad (45)$$

depending on how the sources j are reconnected. In this picture, each light shaded area is simply connected after the average over j has been performed, and all the sources j it contains are linked in all the possible ways that preserve its connectedness.

The first term in the r.h.s. of eq. (45) corresponds to contributions where we connect together only j 's that belong to the same factor \mathbf{S} , $\langle \mathbf{S}(x) \rangle_j \langle \mathbf{S}(y) \rangle_j$. Our previous remark about the average over j of the 1-point function $\Phi(X)$ also applies here to $\langle \mathbf{S}(x) \rangle_j$: its Fourier transform only contains very soft modes of the order of the inverse of the nuclear radius. It is therefore nearly zero for the typical particle momentum $p \sim Q_s$ we are interested in here. Thus only the connected terms in the average of the source term $\langle \mathbf{S}(x)\mathbf{S}(y) \rangle_j$ matter in the kinetic equation.

5.3 Magnitude of field insertions

The source term in eq. (45), as well as the other terms in the right hand side of the Boltzmann equation, involve insertions of the classical field $\phi(x)$. In this subsection, we present a simple power counting that enables us to estimate the magnitude of such insertions. To simplify the discussion, we shall assume that the space-time coordinate X corresponds to sufficiently late times when the external source j is zero and its influence is only felt through the classical field $\phi(X)$ generated by the source at earlier times.

Following the discussion after eq. (34), we assume that there is hard momentum scale Q_s in the problem—the saturation scale in heavy ion collisions. Typical particle momenta are of order $p \sim Q_s$. In our toy model, the coupling constant g has the dimension of a mass in 4 dimensions. To mimic the power

counting in QCD, we will write it as

$$g \equiv \lambda Q_s, \quad (46)$$

where λ , like the QCD coupling constant, is dimensionless. We assume that the coupling constant $\lambda \ll 1$.

To estimate the order of magnitude of the source term given in eq. (45), it is not sufficient to know the magnitude of the classical field. Kinematical phase space constraints can alter the naive power counting. As these considerations will apply equally to the collision terms in the Boltzmann equation, it is worth our while to discuss the power counting for the source term at length here.

From eq. (25) and eq. (45), the naive power counting for the source term would give

$$\mathbf{\Pi}^S(X, p) = \frac{\lambda^2 Q_s^2}{4} \int d^4 s \, e^{ip \cdot s} \left\langle \phi^2(X + \frac{s}{2}) \phi^2(X - \frac{s}{2}) \right\rangle_j. \quad (47)$$

We will demonstrate that eq. (47) vanishes when the momentum carried by the classical field ϕ is nearly on shell. Rewriting this expression entirely in momentum space in terms of the Fourier transform $\tilde{\phi}(k)$ of the classical field,

$$\begin{aligned} \mathbf{\Pi}^S(X, p) &= \frac{\lambda^2 Q_s^2}{4} \int d^4 s \int \frac{d^4 k_1}{(2\pi)^4} \dots \frac{d^4 k_4}{(2\pi)^4} e^{ip \cdot s} \\ &\times e^{-ik_1 \cdot (X + \frac{s}{2})} e^{-ik_2 \cdot (X + \frac{s}{2})} e^{-ik_3 \cdot (X - \frac{s}{2})} e^{-ik_4 \cdot (X - \frac{s}{2})} \left\langle \tilde{\phi}(k_1) \tilde{\phi}(k_2) \tilde{\phi}(k_3) \tilde{\phi}(k_4) \right\rangle_j. \end{aligned} \quad (48)$$

For the sake of simplicity, let us assume that the average over the external source j of the product of four fields factorizes into products of averages of two fields as suggested by the source distribution in eq. (33).

$$\left\langle \tilde{\phi}(k_1) \tilde{\phi}(k_2) \tilde{\phi}(k_3) \tilde{\phi}(k_4) \right\rangle_j = \left\langle \tilde{\phi}(k_1) \tilde{\phi}(k_3) \right\rangle_j \left\langle \tilde{\phi}(k_2) \tilde{\phi}(k_4) \right\rangle_j + \text{other contractions}. \quad (49)$$

For illustrative purposes, we consider only one of the possible contractions corresponding to the connected topology of the second term in the r.h.s. of eq. (45)). It is convenient at this point to denote

$$\mathbf{G}_{\text{cl}}^{-+}(x, y) \equiv \langle \phi(x) \phi(y) \rangle_j, \quad (50)$$

so that one has

$$\left\langle \tilde{\phi}(k_1) \tilde{\phi}(k_3) \right\rangle_j = \int d^4 Y \, e^{i(k_1 + k_3) \cdot Y} \mathbf{G}_{\text{cl}}^{-+} \left(Y, \frac{k_1 - k_3}{2} \right). \quad (51)$$

The definition of the object $\mathbf{G}_{\text{cl}}^{-+}(x, y)$ is identical to the usual definition of the $-+$ component of the Schwinger-Keldysh propagators, except, as the notation suggests, it is constructed from the classical solution of the equations of motion

rather than from the full field operator. Inserting this definition into eq. (48) and keeping only the lowest order ¹⁶ in the gradients in X , one obtains

$$\begin{aligned} \mathbf{\Pi}^s(X, p) = & \frac{\lambda^2 Q_s^2}{4} \int \frac{d^4 k}{(2\pi)^4} \mathbf{G}_{\text{cl}}^{-+}(X, k) \mathbf{G}_{\text{cl}}^{-+}(X, p - k) \\ & + \text{other contractions} . \end{aligned} \quad (52)$$

Note that in this case there is only one other contraction, that leads to the same contribution, thereby transforming the prefactor $1/4$ into a $1/2$. If the time X^0 at which this is evaluated is large compared to $(Q_s)^{-1}$, the classical field that enters in the definition of $\mathbf{G}_{\text{cl}}^{-+}$ is mostly on-shell, and one can write

$$\mathbf{G}_{\text{cl}}^{-+}(X, k) \approx 2\pi\delta(k^2 - m^2) f_{\text{cl}}(X, \mathbf{k}) . \quad (53)$$

By analogy with eq. 28, the distribution $f_{\text{cl}}(X, \mathbf{k})$ can be interpreted as representing the “particle content” of the classical field. As eq. (52) has exactly the structure of a $2 \rightarrow 1$ collision term with on-shell particles of equal mass, it is zero because of energy-momentum conservation.

Therefore, to correctly estimate the magnitude of the source term $\mathbf{\Pi}^s$ when the classical field is weak, one needs to properly account for the slight off-shellness of the field Fourier modes. From the equation of motion

$$\frac{\square + m^2}{Q_s^2} \left(\frac{\phi}{\phi^*} \right) + \frac{1}{2} \left(\frac{\phi}{\phi^*} \right)^2 = 0 , \quad (54)$$

the off-shellness of the classical field comes from its self-interactions. The simplest way to take this off-shellness into account is to use the equation of motion in order to write

$$\tilde{\phi}(k) = \frac{\lambda Q_s}{2} \frac{1}{k^2 - m^2} \int \frac{d^4 q}{(2\pi)^4} \tilde{\phi}(q) \tilde{\phi}(k - q) , \quad (55)$$

and to replace some of the $\tilde{\phi}$'s in eq. (48) by the above relation. It is sufficient to replace two $\tilde{\phi}$'s in order to lift the kinematical constraints that came from the classical field having only nearly on-shell Fourier modes. This substitution is straightforward. One obtains,

$$\begin{aligned} \mathbf{\Pi}^s(X, p) = & \left(\frac{\lambda^2 Q_s^2}{4} \right)^2 \int \frac{d^4 k}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \frac{1}{(k^2 - m^2)^2} \\ & \times \mathbf{G}_{\text{cl}}^{-+}(X, q) \mathbf{G}_{\text{cl}}^{-+}(X, k - q) \mathbf{G}_{\text{cl}}^{-+}(X, p - k) + \text{other contractions} . \end{aligned} \quad (56)$$

¹⁶At this order, this is equivalent to assuming, from the translational invariance in the transverse plane of a large nucleus, that eq. (51) can be replaced by

$$\left\langle \tilde{\phi}(k_1) \tilde{\phi}(k_3) \right\rangle_j \approx (2\pi)^4 \delta(k_1 + k_3) \mathbf{G}_{\text{cl}}^{-+}(X, k_1) .$$

This contribution to the source term can be represented diagrammatically as

$$\Pi^S(X, p) = \text{diagram of a circle with wavy lines and external lines labeled } -, +, -, +, \quad (57)$$

where the solid lines represent ordinary vacuum propagators ($1/(k^2 - m^2)$) and the wavy lines represent the correlation function $\mathbf{G}_{\text{cl}}^{+-}$. It is interesting to note that this contribution is identical in form to what one would have obtained in the collision term of the conventional Boltzmann equation, except that here the G^{+-} propagators are made up of the classical fields.

We are now in a position to estimate the power counting of contributions to the source term. First, the order of magnitude of the denominators $k^2 - m^2$ is Q_s^2 because the momentum transfer k is of order Q_s (and is not particularly close to the mass shell). Each $\mathbf{G}_{\text{cl}}^{-+}$ contains a delta function. Two of them can be used to perform for free the integrations over the energies k^0 and q^0 , while the third provides the value of one angular integration variable. We finally obtain the estimate

$$\Pi^s(X, p) \sim \frac{Q_s^2}{\lambda^2} \left(\frac{n_{\text{cl}}(X)}{n^*} \right)^2 \frac{f_{\text{cl}}(X, \bar{p})}{f^*}, \quad (58)$$

where

$$n_{\text{cl}}(X) \equiv \int \frac{d^3 \mathbf{k}}{(2\pi)^3} f_{\text{cl}}(X, \mathbf{k}) \, , \quad (59)$$

$f^* \equiv \lambda^{-2}$, $n^* \equiv Q_s^3 \lambda^{-2}$ and $n_{\text{cl}}(X)$ is the spatial density of particles corresponding to the classical field. The expressions f^* and n^* correspond respectively to the maximal values of f_{cl} and n_{cl} can have at early times ($\lesssim Q_s^{-1}$). The argument \bar{p} cannot be specified exactly (in fact, eq. (58) is an oversimplified version of the actual formula for Π^S), but it is a momentum whose components are of the same order of magnitude as those of p , the momentum of the produced particle. This is an important point, because as time increases, the support of f_{cl} shrinks in the p_z direction because of the longitudinal expansion of the system, thus making $f_{\text{cl}}(X, \bar{p})$ decrease as well (while in the center of its support, it would stay constant).

Even if eq. (56) is not valid (say, if the average over j were to generate connections among the fields that are not pairwise), the estimate of $\mathbf{\Pi}^s$ one obtains from it has a much wider range of validity. (Eq. (58) is valid even in the saturated regime.) We also note that as $\mathbf{\Pi}^s$ is an inhomogeneous term existing even when $f = 0$, its magnitude depends only on the time dependence of the classical field $\phi(x)$ through f_{cl} and n_{cl} .

5.4 Collision terms

The estimate of the various contributions to the collision term follow very closely that of the source term. Let us start by listing the terms we need to estimate. Because of the presence of the background field and of the average over the

and the corresponding expression reads

$$\begin{aligned} \mathcal{C}_1[f] &= \left(\frac{\lambda^2 Q_s^2}{4} \right)^2 \int \frac{d^4 k}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \frac{1}{(k^2 - m^2)^2} \\ &\quad \times \mathbf{G}^{-+}(X, q) \mathbf{G}_{\text{cl}}^{-+}(X, k - q) \mathbf{G}^{-+}(X, p - k) + \text{other contractions} . \end{aligned} \quad (65)$$

Here we replace two out of three correlators $\mathbf{G}_{\text{cl}}^{-+}$ by \mathbf{G}^{-+} ; the power counting for this diagram is then

$$\mathcal{C}_1[f] \sim \frac{Q_s^2}{\lambda^2} \left[\frac{n_{\text{cl}}(X)}{n^*} \frac{n(X)}{n^*} \frac{f(X, \bar{p})}{f^*} \oplus \left(\frac{n(X)}{n^*} \right)^2 \frac{f_{\text{cl}}(X, \bar{p})}{f^*} \right] . \quad (66)$$

Finally, for the 2-loop contribution to the collision term, we have

$$\mathcal{C}_2[f] = \text{diagram with shaded circle} = \text{diagram with white circle} , \quad (67)$$

$$\begin{aligned} \mathcal{C}_2[f] &= \left(\frac{\lambda^2 Q_s^2}{4} \right)^2 \int \frac{d^4 k}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \frac{1}{(k^2 - m^2)^2} \\ &\quad \times \mathbf{G}^{-+}(X, q) \mathbf{G}^{-+}(X, k - q) \mathbf{G}^{-+}(X, p - k) + \text{other contractions} , \end{aligned} \quad (68)$$

and

$$\mathcal{C}_2[f] \sim \frac{Q_s^2}{\lambda^2} \left(\frac{n(X)}{n^*} \right)^2 \frac{f(X, \bar{p})}{f^*} . \quad (69)$$

5.5 Discussion

Following the power counting in equations (58), (63), (66) and (69), we are now in a position to discuss qualitatively the relative magnitude of the various terms at different stages of the evolution of the system. An important facet of the temporal evolution is that the functions f_{cl} and n_{cl} are determined once and for all from the classical field $\phi(x)$ itself. They do not receive any feedback from the particle distributions or densities, denoted by f and n respectively, that are created in the evolution by the source term $\mathbf{\Pi}^s$. The time dependence of $n_{\text{cl}}(X)$ is driven by the expansion of the system; therefore at times larger than $(Q_s)^{-1}$, one has

$$\frac{n_{\text{cl}}(X)}{n^*} \sim \frac{1}{Q_s \tau} . \quad (70)$$

This reduction of the classical particle density n_{cl} with time happens because the support in momentum space of the corresponding phase-space density f_{cl} shrinks. At a given space-time location X (specified by the space-time rapidity

η), only particles with a matching momentum rapidity $y = \eta$ can stay for a long time. Therefore, inside its support, f_{cl} remains constant satisfying

$$\frac{f_{\text{cl}}(y \approx \eta)}{f^*} \sim 1. \quad (71)$$

Note that at times smaller than $(Q_s)^{-1}$, f_{cl}/f^* and $n_{\text{cl}}(X)/n^*$ are also both of order 1 because the classical field is completely saturated.

However, in all the estimates of the previous subsection, f_{cl} is evaluated at some arbitrary location X and momentum \bar{p} . Therefore, \bar{p} will eventually fall outside of the support of f_{cl} , and f_{cl} will decrease quickly¹⁹ after that happens. For f_{cl} , which comes entirely from the classical field ϕ , the only time-scale in the problem is $1/Q_s$ and thus we expect f_{cl} to start decreasing at times larger than $1/Q_s$.

At early times, $\tau \rightarrow 0$, the system does not have particles yet and we have $f = n = 0$. Obviously, in this regime, only the source term $\mathbf{\Pi}^s$ is important in the right hand side of the Boltzmann equation. The corresponding physics is that a population of particles, described by the occupation number f , is built up from the decay of the classical field. However, these particles are still too few to have collisions at a significant rate. Eq. (58) tells us that $\mathbf{\Pi}^s \sim \frac{Q_s^2}{\lambda^2}$ in this regime.

As a rough estimate, if we integrate this source term in the range $0 \leq \tau \leq Q_s^{-1}$, we find that the occupation number for particles of momentum $p \sim Q_s$ at a time $\tau \sim Q_s^{-1}$ is

$$\frac{f(\tau = Q_s^{-1})}{f^*} \sim 1. \quad (72)$$

At this time, all the components of the momenta of these particles are typically of order Q_s . Therefore, we also have

$$\frac{n(\tau = Q_s^{-1})}{n^*} \sim 1. \quad (73)$$

At times around $(Q_s)^{-1}$ all the terms in the right side of the Boltzmann equation are of equal magnitude. Indeed, in this regime, terms with an arbitrarily large number of loops contribute equally to the collision term when $f \sim f^*$. There would therefore be an equally large $\mathcal{C}_3[f]$, $\mathcal{C}_4[f]$, etc... In practice, this means that one should start using the Boltzmann equation only at later times.

At later times, $\tau \geq Q_s^{-1}$, collisions among the particles become important and their qualitative effect is to broaden the momentum distribution of the particles represented by f , thereby counteracting the effect of the expansion²⁰ of the system. Thanks to these collisions, $f(X, \bar{p})$ falls at a lesser rate compared to $f_{\text{cl}}(X, \bar{p})$ (which is not affected by collisions), which eventually leads to the

¹⁹The precise time dependence of this fall depends on the p_z dependence of f_{cl} . To take an extreme case, there would be no fall at all if f_{cl} is independent of p_z .

²⁰In the absence of collisions, f would be affected by the system expansion in a similar way to f_{cl} , and its support would shrink like τ^{-1} in the p_z direction.

dominance of $\mathcal{C}_2[f]$ over all the other terms in the right hand side of the Boltzmann equation. When this occurs, our Boltzmann equation is identical to the usual one. The detailed mechanisms of this transition between the classical field dominated regime and the kinetic regime will be discussed in a future work. In particular, it will be interesting to compare, for the QCD case, the temporal evolution of the kinetic equation for the glasma with the “bottom up” scenario of thermalization [41].

6 Summary and Outlook

In this work, we developed the formalism of Refs. [1,2] for particle production in the presence of strong sources to construct a kinetic theory relevant for the early “glasma” stage of a heavy ion collision. In particular, we considered for simplicity, the dynamics of a ϕ^3 theory in the presence of strong sources. Much of our discussion however is completely general and could in principle be extended to describe the dynamics of gauge fields exploding into the vacuum after a heavy ion collision. We showed that the relevant kinetic equation for the particle distributions f has the structure of a Boltzmann equation with an additional inhomogeneous (f -independent) source term denoting particle creation from the decay of the classical field. The collision terms in the Boltzmann equation also have novel features. In addition to the usual contribution from the two loop self energy, there are 0-loop and 1-loop contributions that affect the particle phase space distributions. We outlined the power counting that controls the magnitude of the contributions of the source term and the collision terms. The temporal evolution of these contributions was discussed only briefly and will be discussed in detail elsewhere.

There are several unresolved issues that should be addressed in future work. Primarily, we would like to understand precisely how the derivation here plays out in the QCD case. In Refs. [42–45], it was shown that instabilities of the Weibel type [46–51] can spoil the bottom up scenario of thermalization. Such an instability is also seen in the CGC framework in the explosive growth of small fluctuations about the classical background field [52–54] and has a natural interpretation as quantum fluctuations about the classical background fields on the light cone [55]. A numerical study of instabilities in a field+particle framework has been performed [56] but we would like to better understand how the effects of such instabilities manifest themselves in the kinetic equation for the glasma. It would be especially interesting to uncover whether Kolmogorov turbulent spectra [57] arise as a consequence of these instabilities [58,59] and whether this phenomenon of “turbulent thermalization” can be accommodated in our kinetic framework.

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A Average over the sources

We have seen that it is crucial for the validity of the gradient expansion to consider quantities averaged over the source j coupled to the fields. We shall discuss briefly here how this average can be accounted for in our formalism.

Let us start from the generating functional for Green's functions of the Schwinger-Keldysh formalism²¹ $Z_j[\boldsymbol{\eta}]$, for a given configuration j of the external source. We define it in such a way that the n -point Green's functions is obtained by differentiating n times with respect to $\boldsymbol{\eta}$, and then by setting the auxiliary source $\boldsymbol{\eta}$ to zero. From what we have said in section 2, this generating functional is related to the sum of all the vacuum-vacuum diagrams by :

$$Z_j[\boldsymbol{\eta}] = e^{i\mathcal{V}_{SK}[j+\boldsymbol{\eta}]} , \quad (74)$$

where we have again used a compact notation compared to eq. (9). We do not use a boldface letter for the external source j , in order to emphasize the fact that it is identical on both branches of the closed time path.

From this object, it is very easy to construct the generating functional for Green's functions that are averaged over some ensemble of external sources, with a distribution $W[j]$, as :

$$Z[\boldsymbol{\eta}] = \int [Dj] W[j] e^{i\mathcal{V}_{SK}[j+\boldsymbol{\eta}]} . \quad (75)$$

In order to see how this average over j can be accounted for in the Feynman rules, it is useful to write the generating functional for a fixed j as follows :

$$\begin{aligned} e^{i\mathcal{V}_{SK}[j+\boldsymbol{\eta}]} &= \exp \left(i \int_{\mathcal{C}} d^4x V \left(\frac{\delta}{\delta \boldsymbol{\eta}(x)} \right) \right) \\ &\times \exp \left(-\frac{1}{2} \int_{\mathcal{C}} d^4x d^4y (j(x) + \boldsymbol{\eta}(x)) \mathbf{G}^0(x, y) (j(y) + \boldsymbol{\eta}(y)) \right) , \end{aligned} \quad (76)$$

where V is the sum of all the interaction terms in the theory under consideration (i.e. all the terms of the Lagrangian density that are of degree ≥ 3 in the field). In this formula, $\mathbf{G}^0(x, y)$ denotes the free propagator in the Schwinger-Keldysh

²¹In order to keep the notations compact, we denote by a boldface letter $\boldsymbol{\eta}$ the pair $\boldsymbol{\eta} \equiv (\eta_+, \eta_-)$, where the \pm indices refer to the Schwinger-Keldysh closed time path.

formalism (as opposed to the full propagator defined in eq. (9)). It is now convenient to write the second exponential in the r.h.s. of eq. (76) as the action of a translation operator on a functional that does not depend on j ,

$$\begin{aligned} \exp\left(-\frac{1}{2}\int_{\mathcal{C}} d^4x d^4y (j(x) + \boldsymbol{\eta}(x))\mathbf{G}^0(x, y)(j(y) + \boldsymbol{\eta}(y))\right) = \\ = \exp\left(i\int_{\mathcal{C}} d^4z j(z)\frac{\delta}{\delta\boldsymbol{\eta}(z)}\right) \exp\left(-\frac{1}{2}\int_{\mathcal{C}} d^4x d^4y \boldsymbol{\eta}(x)\mathbf{G}^0(x, y)\boldsymbol{\eta}(y)\right). \end{aligned} \quad (77)$$

By inserting this formula in eq. (76), and then in eq. (75), we obtain the following expression :

$$\begin{aligned} Z[\boldsymbol{\eta}] = \left\{ \int [Dj] W[j] e^{i\int_{\mathcal{C}} d^4z j(z)\frac{\delta}{\delta\boldsymbol{\eta}(z)}} \right\} \\ \times \exp\left(i\int_{\mathcal{C}} d^4x V\left(\frac{\delta}{\delta\boldsymbol{\eta}(x)}\right)\right) \exp\left(-\frac{1}{2}\int_{\mathcal{C}} d^4x d^4y \boldsymbol{\eta}(x)\mathbf{G}^0(x, y)\boldsymbol{\eta}(y)\right). \end{aligned} \quad (78)$$

The terms on the second line are nothing but the generating functional for the same theory *without any external source* (since it does not depend on j). As we can see, the effect of the average over the external source j is to bring a prefactor which is a certain functional of the operator $\delta/\delta\boldsymbol{\eta}$. Such a term can be interpreted as additional couplings among the fields, since one can always write :

$$\left\{ \int [Dj] W[j] e^{i\int_{\mathcal{C}} d^4z j(z)\frac{\delta}{\delta\boldsymbol{\eta}(z)}} \right\} \equiv \exp\left(i\int_{\mathcal{C}} d^4x U\left(\frac{\delta}{\delta\boldsymbol{\eta}(x)}\right)\right). \quad (79)$$

What this derivation makes obvious is that, for calculating averaged quantities over the ensemble of external sources j , one can forget the external sources altogether, and include additional vertices to the theory²², as prescribed by eq. (79). Note that this is equivalent to calculating a quantity in an arbitrary j , and then reconnecting all the j 's among themselves in all the possible ways permitted by $\ln(W[j])$.

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²²Note that formally, the “potential” U is the connected part of the Fourier transform of the functional $W[j]$. This means that in the particular case where $W[j]$ is a Gaussian, there is only one coupling in the new potential U , which couples two fields. Because “interaction terms” that are quadratic in the fields can in general be handled in closed form, it is possible in this case to absorb the potential U in a modification of the propagator $\mathbf{G}^0(x, y)$ in eq. (78).

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